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DEVELOPING A MULTIPLICATIVE MATHEMATICAL MODEL FOR GEOMECHANICAL STABILITY OF TAILINGS DAMS BY SUCCESSIVE APPROXIMATION METHOD

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Abstract. The issue of developing the model-based tools for monitoring, assessing, and forecasting safe natural and technological states of hydraulic structures, including slope structures of tailings dams, is one of the topical problems. The models represented in regulatory documents that establish requirements for the design, construction, and operation of hydraulic structures are primarily based on semi-empirical limit state models for calculating permissible stability parameters of dams. At the same time, modern international practice shows that in the design and operation of hydraulic structures, models based on the theory of deformable solid bodies combined with the finite element method are most commonly used.

The purpose of this work is to develop a multiplicative mathematical model for determining the sensitivity of the dam's stability function to variations in its structural parameters and the physical-mechanical properties of its components using the method of successive iterative approximation.

In the course of the study, during model development, the approximation of the stability coefficient was carried out in a multiplicative form, where the components of the product are power functions, each depending only on a single parameter. The sensitivity to parameter variation was determined by the exponent indicators of these functions. The approximation coefficient, or multiplier, served as a free parameter that regulated the adequacy of the model parameters at the forecast point during the extrapolation procedure. The input data for obtaining the approximation model of the stability factor of slope structures were generated through a series of numerical experiments based on the geomechanical characteristics of a real object – the internal dam of a tailings storage facility.

The scientific results of the study are as follows: the construction of a model for the stability safety factor based on the method of successive approximation (SAM), which made it possible not only to obtain an analytical form of the criterion in the neighbourhood of a point but also to extend the solution to the entire domain of the function, with errors not exceeding values acceptable for applied geomechanics problems; and the formulation of a hypothesis regarding the availability of a representation of functions in the form of a product of functions, each depending on a single parameter. It has been established that through the synthesis of the SAM and computer-based experimental studies, it is possible to obtain families of deterministic multiplicative mathematical models of various types of objects.

The practical results of the study include a derived formula for determining a stability safety factor of slope systems in tailings dam structures, which allows for an approximate assessment of the risks of stability loss due to variations in parameter values.

Keywords: numerical experiment, sensitivity theory, variation of the parameters, approximation of a function, approximate evaluation, low error, tailings storage facility, geomechanical stability of the dam.

1. Introduction

Storage facilities for wet enrichment or production waste (tailings, slag, ash, and sludge storage facilities) are hazardous, high-risk hydraulic structures. The main structural elements ensuring the safe operation of a tailings storage facility are the protective external and internal dams. The external dams ensure technogenic safety for the environment, while the internal dams primarily perform technological functions [1]. The safety of a dam during its exploitation period is determined by the geomechanical stability of the structure's slope systems. Currently, within the operational and on-site monitoring systems of tailings storage facilities, there are practically no universally accepted methods for assessing and forecasting the stability condition of slope systems. Despite numerous recommendations in regulatory documents concerning the monitoring and operation of dams, there is no standard document that defines methods for calculating permissible loads and deformations in hydraulic structures. At the same time, the performance of technological, comprehensive monitoring activities and preliminary scientific and engineering assessments requires fast-

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acting, compact models for the real-time assessment of the condition of slope structures, which underlines the relevance and importance of this research direction.

A milestone in conducting experimental research is choosing an appropriate mathematical model to describe the study process. There are relatively few types of analytical models: additive or multiplicative. The multiplicative form of the model is the most widely used. This form is the most convenient because the components of the product can be functions of various types. However, when selecting a multiplicative model, there is one significant difficulty, i.e., the uncertainty of the approximation coefficient, or multiplier. Due to the unclear nature of its determination, this coefficient is sometimes referred to as the “coefficient of ignorance”. The adequacy of the model in representing the real process; and even more importantly, the adequacy of the parameters at the forecast point during extrapolation depends on its correct selection. To determine the optimal operating modes of technical systems and to assess the risks of exceeding permissible performance limits, it is necessary to evaluate the influence of individual parameters on the quality function of their operation.

Thus, for deterministic mathematical models (MM), the influence of parameters can be determined using the methods of sensitivity theory (ST). However, due to the considerable complexity of such analyses, sensitivity theory has not yet found wide application in engineering practice. We propose to apply the approximation of the quality criterion (instead of using ST directly) to determine sensitivity in technical applications. If the approximation of the selected criterion is performed in multiplicative form, where the components of the product are power functions, each depending on a single parameter, then the exponents can be used to approximately determine the sensitivity of the criterion to the parameter variations. The greater the exponent, the stronger the influence of that parameter on the criterion. Thus, it becomes possible not only to obtain an approximate assessment of the parameters’ influence on the criterion itself but also to make conclusions about the risk of the system exceeding its permissible limits. The successful experience of applying the successive approximation method (SAM) in the problems of applied mechanics makes it possible not only to obtain the analytical form of a criterion in the neighbourhood of a point but also to extend the solution across the entire domain of the function. The errors associated with such an extension generally do not exceed 5–7%, which is sufficient for most applied geomechanical problems. The accuracy of the criterion determination can be increased to the required level by narrowing the range of parameter variation, i.e., by limiting the intervals of parameter changes.

2. Methods

As a base method for calculating the geomechanical stability of slope systems in hydraulic structures, one of the classical limit equilibrium methods can be proposed, such as those developed by Fellenius, Janbu, Bishop, Fisenko, and others [2,3], or more precise methods based on the theory of deformable solid bodies [4-6]. Currently, with the advancement of computational tools, classical limit equilibrium methods are increasingly being replaced by or combined with higher-accuracy methods.

In this study, we selected a base method of higher accuracy, combined with the finite element method (FEM). The effectiveness of such methods for dam stability calculations has been confirmed by numerous recent studies [7-10].

The mathematical model for the study should ensure a certain level of accuracy while approximating of the detailed large-scale models of complex systems. Therefore, the idea emerged to construct a simplified multiplicative model for determining the safety factor of a slope system in a hydraulic structure. This model combines the theory of deformable solid bodies, solved using FEM, with the SAM and sensitivity theory.

The goal of the work is to develop an analytical formula to determine the sensitivity of the dam's stability function to the variations in its structural parameters and the physical-mechanical properties of its components, using SAM.

3. Theoretical part

One of the major challenges in studying complex systems is to obtain reliable models of parameters $\{x_1, x_2, ..., x_q\}$ that describe the surface of performance characteristics $\{y_1, y_2, ..., y_q\}$.. An active experiment implies the ability to actively influence the process being studied according to a pre-established plan. Active experiments can be conducted on both physical and mathematical models. Most often, such models are represented as a black box [11] (see Fig. 1).

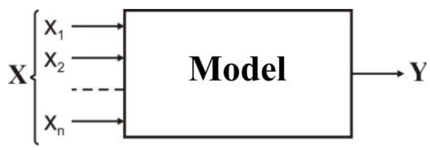


Figure1 - Diagram of the black box operation

To study the influence of the parameters of multifactor processes on the performance indicators, multiplicative models of type (1) have been widely used

$$Y = \alpha \prod_{i=1}^{n} f_i(x_i),$$

where Y is the outcome variable; $f_i(x_i)$ are some functions; x_i are process variables; n is dimension of the outcome variable space; and α is "ignorance coefficient".

The advantages of such a model representation include a significant reduction in the number of experiments required for its development. Indeed, instead of conducting experiments on a mesh of parameters, experiments must be performed only along the lines formed by the intersection of the performance function surface with coordinate planes parallel to the axes, passing through a certain point within the domain. Moreover, multifactorial models allow for the accumulation of information as the number of factors increases. This accumulation is achieved by changing and introducing additional functions of the required factors. The previously collected information in the form of single-factor dependencies remains in the process model. Despite these advantages, three significant drawbacks limit the widespread use of such model representations:

1. The class of product functions must be defined based on the information available before conducting the experiment.

2. The procedure for determining coefficient α, often referred to as the "coefficient of ignorance," remains undefined. It is most commonly found experimentally, based on the equality of the resulting characteristic with a known value.

3. The error evaluation of the presented model is performed only by comparing it with the experimental data and cannot be predicted in advance. This, in turn, leads to uncertainty regarding the step sizes for the parameter variations, and thus the number of experiments required.

Attempts to determine the "coefficient of ignorance" using the proposed formulas have been unsuccessful [12-15]. It became clear that, for the correct determination of this coefficient, it is necessary not to assign the form of the formula in advance, but to derive its expression through the application of mathematical methods.

The importance of such representations of experimental data arises from the fact that experiments can also include computational experiments. For example, the availability of software packages with advanced user interfaces based on the FEM and boundary element methods (BEM) has significantly expanded the range of practical problems that can be modelled and analysed. The results of studies on the processes using numerical methods often provide functions of the required parameters, presented in a tabular form. Based on these tables, graphical dependencies of the function on any parameter are constructed. However, the problem of assessing the impact of parameters on the characteristics of the stress-strain state cannot be solved within the scope of these software packages and remains a separate and complex problem.

For deterministic MM that are described by systems of differential equations (DEs) and have analytical solutions, the influence of parameters can be determined using sensitivity theory (ST) methods [12]. The part of ST related to studying the effect of parameter changes on the system's characteristics is commonly called parametric sensitivity theory. In the sequel, the term sensitivity will refer to parametric sensitivity [12].

Let's consider the process described by a system of differential equations. The search for a complex control function proceeded according to the following scheme:

1) solve the system of differential equations and select the necessary solutions using a specific criterion;

2) investigate the continuity and stability of the solutions with respect to the parameter variations;

3) obtain the sensitivity equations (a generalized derivative of the selected solution to the initial system of differential equations generates the system of sensitivity equations);

4) find the solutions (the sensitivity functions) of the sensitivity equations;

5) study the stability of the sensitivity functions with respect to the parameter variations; and

6) select from the set of sensitivity functions the ones that ensure the correct control of the system.

The represented scheme for selecting a control system for the process is not simple and requires significant effort, qualifications, and time to implement. Given the significant advantages of the identified control systems, incremental application of ST has expanded in the field of technical applications, including tasks related to the mechanics of continuous media.

SAM allows for the approximate determination not only of the analytical form of the model but also for assessing the response speed of the selected criterion to the parameter variations. This enables an approximate evaluation of the influence of process parameters [18,19].

For example, a comparison of the results from solving a classical problem of determining the stress-strain state near a circular cross-section tunnel reinforced with anchors using FEM in a mass of rock showed promising results [20].

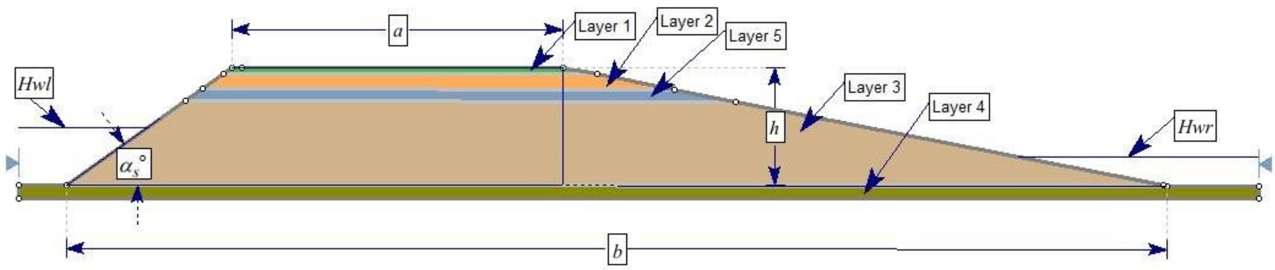
The use of SAM for various problems of geotechnical mechanics has demonstrated its effectiveness in obtaining an approximate representation of functions in the neighbourhood of a fixed point within the domain [6]. However, as practical experience with applying SAM to mechanical problems shows, the required functions exhibit sufficient accuracy for engineering calculations across the entire parameter domain. The relative error increases as the solution approaches the boundary of the parameter definition range, but for most problems, it did not exceed 10%, which is entirely acceptable for geotechnical engineering calculations. Solutions of the problems of geotechnical mechanics using FEM methods are smooth, as well as the functions that describe the stress state of the object.

Thus, the need arose to expand the application of SAM and attempt to use it for modelling the stress-strain state in FEM-based problems of geotechnical mechanics.

The object of study is the tailings dam of heterogeneous construction (comprising five horizontal layers of rock with different physical-mechanical properties), asymmetric shape (with dam slopes varying from gentle to steep), and subjected to hydrostatic loading [21] (see Figure 2).

The task is solved in the context of plane strain formulation [4, 5].

Successful use of SAM in the problems of applied mechanics [22-24] demonstrates that, although the function is determined around a fixed point, solutions to practical problems can be extended across the entire domain of the function. The errors in such a representation increase as the solution approaches the boundary of the domain, but they do not exceed 5–7%. This level of accuracy is satisfactory for engineering calculations in geotechnical mechanics, as the input data for these calculations are determined with the same level of precision. The accuracy can be increased to the necessary level by narrowing the domain of parameter variation. Successful application of SAM in practical applications has allowed for the generalization of the research results and the formulation of a hypothesis about the availability of such a representation for a wider range of problems.



α_s is slope angle, degree; a is crest width, m; b is base width, m; h is dam height, m; H_{wl}= 7 is water level on the left, m; H_{wr}= 4 is water level on the right, m

Figure 2 - Scheme of the object model and its parameters

HYPOTHESIS: Let there exist scalar function $F(X)=F(x_1,x_2,...,x_n)$ that is bounded, well-defined, and continuous in a closed region $\overline{D}$ of scalar field P. Then, for $\forall M \in D; \forall \varepsilon \geq 0 \exists U_\varepsilon(M) \subset \overline{D}$ in the neighbourhood of point $M_0\left(x_1^0, x_2^0, x_3^0, ... x_n^0\right)$, function $F(X)$ can be represented in the form:

$$\left|F(X)-\phi(X)\right| \leq \varepsilon, \forall M_0 \in U_\varepsilon(M_0), \quad (1)$$

Where $U_\varepsilon(M_0)$ is neighbourhood of the function;

$$\phi(X)=\alpha\prod_{i=1}^{n}\varphi_i(x_i)=\prod_{i=1}^{n}\alpha_i g_i(x_i), \quad (2)$$

where α is coefficient of approximation; $g_i(x_i)$ is functions of approximation for functions $\varphi_1(x_1), \varphi_2(x_2), \varphi_3(x_3)...\varphi_n(x_n)$; and α_i is coefficients of approximation for functions $g_i(x_i)$. Functions φ_i are defined as follows:

$$\begin{cases} \varphi_1(x_1)=F\left(x_1,x_2^0,x_3^0,...,x_n^0\right); \\ \varphi_2(x_2)=F\left(x_1^0,x_2,x_3^0,...,x_n^0\right); \\ \varphi_3(x_3)=F\left(x_1^0,x_2^0,x_3,...x_n^0\right); \\ ... \\ \varphi_n(x_n)=F\left(x_1^0,x_2^0,x_3^0,...x_n\right). \end{cases} \quad (3)$$

and α is coefficient of approximation determined according to the formula:

$$\alpha = \frac{F\left(x_1^0, x_2^0, x_3^0, \ldots, x_n^0\right)}{\prod_i^n \alpha_i g_i\left(x_i^0\right)}. \qquad (4)$$

As evidenced by the experience of using the indicated approach of representing function $F(X) = F(x_1, x_2, x_3, \ldots x_n)$ in the neighbourhood of point $M_0(x_1^0, x_2^0, x_3^0, \ldots x_n^0)$, for a significant number of problems in geotechnical mechanics, it provides sufficient accuracy for engineering calculations over the entire domain of $\overline{D}$.

The algorithm for applying SAM can be presented as a sequence of the following steps [13] (see Fig. 2):

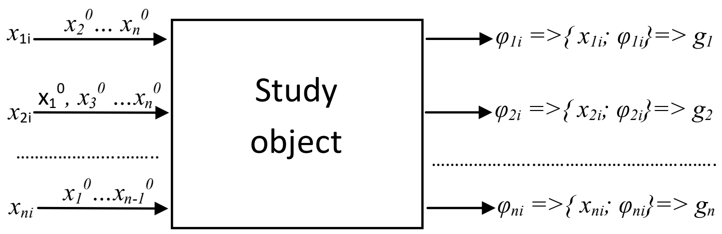


Figure 3 - Explanation of the algorithm application

Step 1. Select a point from the domain of the function

$$M = M\left(x_1^0, x_2^0, x_3^0, \ldots, x_n^0\right), M \in \overline{D}; M_0 = M\left(x_1^0, x_2^0, x_3^0, \ldots, x_n^0\right), \qquad (5)$$

where n is number of variables under consideration.

The selection depends on the qualifications of the researcher, or, in case of difficulties with its selection, it is chosen using a simplified procedure at the centre of the definition intervals.

Step 2. Define functions $\varphi_1(x_1), \varphi_2(x_2), \ldots \varphi_n(x_n)$: $\varphi_1(x_1) = F\left(x_1, x_2^0, \ldots, x_n^0\right)$, and $\varphi_2(x_2) = F\left(x_1^0, x_2, x_3^0, \ldots x_n^0\right) \ldots \varphi_n(x_n) = F\left(x_1^0, x_2^0, \ldots x_n\right)$.

In other words, for function $\varphi_1(x_1) = F\left(x_1, x_2^0, x_3^0, \ldots, x_n^0\right)$, only parameter x_1 is variable, and all other coordinated or parameters remain constant, i.e., fixed at point $M_0 = M\left(x_1, x_2^0, x_3^0, \ldots, x_n^0\right)$. It is similar for all functions $\varphi_i(x_1, \ldots, x_n)$, where n is number of variables under consideration.

Step 3. Determine the form of functions $g_1(x_1), g_2(x_2), g_3(x_3), \ldots g_n(x_n)$, which are approximations for functions $\varphi_i(x_i)$.

Functions $g_1(x_1), g_2(x_2), g_3(x_3), \ldots g_n(x_n)$ belong to the class of elementary functions.

Step 4. Determine $\varphi_i(x_i)$ according to the hypothesis, namely:

$$\phi(X)=\alpha\prod_{i=1}^{n}g_i(x_i), \tag{6}$$

i.e., according to the formula (6):

$$\begin{cases}\phi_1(x_1)=\alpha_1 g_1(x_1);\\ \phi_2(x_1,x_2)=\alpha_2 g_1(x_1)g_2(x_2);\\ \dots\\ \phi_n(x_1,x_2,x_3)=\alpha_n g_1(x_1)g_2(x_2)\dots g_n(x_n).\end{cases}, \tag{7}$$

where $\alpha_1,\alpha_3,\dots\alpha_n$ are coefficients of approximations.

Step 5. Define the function in the neighbourhood of point M_0 from the equality

$$F(x_1,x_2,x_3,\dots x_n)\approx\phi(x_1,x_2,x_3,\dots x_n). \tag{8}$$

Thus, we obtain the required representation of the function

$$F(x_1,x_2,x_3,\dots,x_n)\approx\varphi(x_1,x_2,x_3,\dots,x_n)=\alpha_n g_1(x_1)g_2(x_2)\dots g_n(x_n). \tag{9}$$

The location of point $M_0=M\left(x_1^0,x_2^0,\dots,x_n^0\right),\ M\in\overline{D}$ in the domain significantly depends on its topology and therefore affects the way it is represented. The selection of the point in the domain depends on prior knowledge of its characteristics and qualifications of the researcher. In case of complex functions and no prior knowledge about the behaviour of the response function, it is suggested to select the point in the centre of the domain, i.e., to determine the coordinates using the formula:

$$x_i^0=\frac{b_i-a_i}{2}, \tag{10}$$

where a_i and b_i represent the start and end of the interval of the parameter x_i variation. Thus, we find the average value within the interval.

For our case, the intervals of variation are given in the Table 1.

The problem of calculating the stability coefficients of a dam subjected to hydrostatic pressure on its slopes is an important task in the field of land reclamation and water storage. To solve such problems, there are specialized software products, most of which are based on FEM. Without such mathematical tools, it is almost impossible to calculate the stability coefficients. To perform operational analysis of the dam's state, it would be ideal to estimate the risks of stability loss due to parameter variations and determine their effect on the stability coefficient K_{st}.

Table 1 - Intervals of value variations

Variables X_i	Slope angle α_s, degree	Crest width *a*, m	Base width *b*, m	Dam height *h*, m	Unit weight of a layer γ, kN/m³	Angle of internal layer friction θ, degree	Cohesion of the layer material *c*, kPa	Modulus of elasticity *E*, kPa
Minimum value a_i	15	5	50	10	15	11.4	5.4	7800
Maximum value b_i	60	50	140	46	21.3	27.6	51.2	25400
M_0	37.5	27.5	95	28	18.15	19.15	28.3	16600

This problem can be solved using the SAM described earlier. For convenience, we will use the following algorithm.

The first step in the algorithm is to select a base point around which, as a result of successive actions in the algorithm, the analytical formula will be derived. This formula relates the stability coefficient as a function of the parameters, which characterize both the geometry of the dam and the other parameters needed to calculate its value. Figure 2 shows the main structural elements and parameters of the model describing the dam.

The parameters that define the magnitude of the stability coefficient as a function of other parameters and characterize the base point are represented as follows:

$$K_{st} = K\left(\alpha_s{}^0, a^0, b^0, h^0, \gamma^0, \theta^0, c^0, E^0\right). \quad (11)$$

Values of the output data are shown in Table 2.

Table 2 - Values of the output data of the base point

Parameters	$\alpha_s{}^0$	a^0	b^0	h^0	γ^0	θ^0	c^0	E^0
Values	37.5	27.5	95	28	18.15	19.5	28.3	16600

Taking into account that the model under study consists of 5 layers of rocks with different properties, the values of γ^0, θ^0, c^0, and E^0 were determined as the arithmetic mean of the corresponding physicomechanical properties of the lithological components of the rock mass (see Fig. 2):

$$\gamma^0 = \frac{\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5}{5};\ \theta^0 = \frac{\theta_1+\theta_2+\theta_3+\theta_4+\theta_5}{5};$$
$$c^0 = \frac{c_1+c_2+c_3+c_4+c_5}{5};\ E^0 = \frac{E_1+E_2+E_3+E_4+E_5}{5} \quad (12)$$

The stability coefficient of the dam was calculated using specialized software packages based on the finite element method. By conducting a series of calculations of the stability reserve coefficients of the dam slopes with variations in all parameters, a set of results was obtained, which subsequently served as input for the process of successive approximation. Tables 3 and 4 show the most representative examples of calculation results K_{st} from the variation of the geometric parameter of dam height *h* and the geomechanical parameter of cohesion *c*. The influence of variations in other parameters of the object under study has a similar structure.

Table 3 Results of calculating the change in the safety factor of the dam slopes depending on the variation of the dam height

Nv	1	2	3	4	5	6	7	8	9	10
h, m	10	14	18	22	26	30	34	38	42	46
K_{stL}	2.24	1.57	1.25	1.11	0.97	0.81	0.67	0.61	0.6	0.58
K_{stR}	3.89	2.49	1.41	1.12	0.98	0.79	0.63	0.54	0.43	0.36

Table 4. Results of calculating the change in the safety factor of the dam slopes depending on the variation of the material cohesion of the dam layers c, kPa.

Nv	c_1	c_2	c_3	c_4	c_5	c^0	K_{stL}
1	5	5	10	5	2	5.4	1.21
2	8.3333	8.8888	15.5555	15.5555	4.1111	10.4888	1.47
3	11.6666	12.7776	21.111	26.1111	6.2222	15.5777	1.72
4	14.9999	16.6664	26.6665	36.6667	8.3333	20.6666	1.97
5	18.3332	20.5552	32.222	47.2223	10.4444	25.7554	2.23
6	21.6665	24.444	37.7775	57.7779	12.5555	30.8443	2.46
7	24.9998	28.3328	43.333	68.3335	14.6666	35.9331	2.71
8	28.3331	32.2216	48.8885	78.8891	16.7777	41.022	2.95
9	31.6664	36.1104	54.444	89.4447	18.8888	46.1109	3.21
10	34.9997	39.9992	59.9995	100.0003	20.9999	51.1997	3.46

The symbols used in Tables 3 and 4 are as follows: Nv is the variant number; K_{stL} and K_{stR} are safety factors for the left and right slopes, respectively; and $c_1,...,c_5$ are the cohesion values for the respective layers of the rock mass.

4. Results and discussion

Perform the MM procedure using the selected parameters and following the hypothesis procedure.

Step 1. Obtain a set of function values using the set of formulas (3) $\varphi_1(x_1), \varphi_2(x_2) ... \varphi_n(x_n)$.

In other words, for function $\varphi_1(x_1) = F\left(x_1, x_2^0, x_3^0, ..., x_n^0\right)$, the only variable is parameter x_1, while all other coordinates or parameters remain constant, i.e., they are at point $M_0 = M\left(x_1, x_2^0, x_3^0, ..., x_n^0\right)$. It is similar for all functions

$\varphi_2(x_2),\varphi_3(x_3)...\varphi_n(x_n)$, where n is the number of parameters being considered. The mentioned data set is obtained using FEM with the step 2 recommendations of the hypothesis.

$$\begin{cases}\varphi_1(x_1)=F\left(x_1,x_2^0,x_3^0,...,x_n^0\right);\\ \varphi_2(x_2)=F\left(x_1^0,x_2,x_3^0,...,x_n^0\right);\\ ...\\ \varphi_n(x_n)=F\left(x_1^0,x_2^0,x_3^0,...,x_n\right).\end{cases} \quad (13)$$

Step 2. This step is considered completed since we already have functions $\varphi_i(x_i)$.

Functions $\varphi_1(\alpha_s),\varphi_2(a),\varphi_3(b),\varphi_4(h),\varphi_5(\gamma),\varphi_6(\theta),\varphi_7(c)$, and $\varphi_8(E)$ were determined from the variations of the dependent parameters as approximation formulas to calculate the safety factor K_{st}

$$\begin{cases}\varphi_1(\alpha_s)=K\left(\alpha_s,a^0,b^0,h^0,\gamma^0,\theta^0,c^0,e^0\right);\\ \varphi_2(a)=K\left(\alpha_s^0,a,b^0,h^0,\gamma^0,\theta^0,c^0,E^0\right);\\ \varphi_3(b)=K\left(\alpha_s^0,a^0,b,h^0,\gamma^0,\theta^0,c^0,E^0\right);\\ \varphi_4(h)=K\left(\alpha_s^0,a^0,b^0,h,\gamma^0,\theta^0,c^0,E^0\right);\\ \varphi_5(\gamma)=K\left(\alpha_s^0,a^0,b^0,h^0,\gamma,\theta^0,c^0,E^0\right);\\ \varphi_6(\theta)=K\left(\alpha_s^0,a^0,b^0,h^0,\gamma^0,\theta,c^0,E^0\right)\\ \varphi_7(c)=K\left(\alpha_s^0,a,b^0,h^0,\gamma^0,\theta^0,c,E^0\right)\\ \varphi_8(E)=K\left(\alpha_s^0,a,b^0,h^0,\gamma^0,\theta^0,c^0,E\right).\end{cases} \quad (14)$$

Step 3. Set functions $g_1(x_1),g_2(x_2),g_3(x_3),...g_n(x_n)$, being the approximations for function $\varphi_i(x_i)$, as power functions.

It should be noted that representing the approximating function as a power function can be justified by the small range of the original function, the use of the root-mean-square deviation procedure, and the small errors in the absolute values of the original function.

$$\begin{cases} g_1(\alpha_s) = K(\alpha_s, a_0, b_0, h_0, \gamma_0, \theta_0, c_0, e_0) = \dfrac{16.7}{\alpha_s^{\,0.61557}}; \\ g_2(a) = K(\alpha_{s0}, a, b_0, h_0, \gamma_0, \theta_0, c_0, e_0) = \dfrac{2.159}{a^{0.0054}}; \\ g_3(b) = K(\alpha_{s0}, a_0, b, h_0, \gamma_0, \theta_0, c_0, e_0) = 1.07779 b^{0.120867}; \\ g_4(h) = K(\alpha_{s0}, a_0, b_0, h, \gamma_0, \theta_0, c_0, e_0) = \dfrac{18.4654}{h^{0.92166}}; \\ g_5(\gamma) = K(\alpha_{s0}, a_0, b_0, h_0, \gamma, \theta, c_0, e_0) = \dfrac{15.15}{\gamma^{0.6685}}; \\ g_6(\theta) = K(\alpha_{s0}, a_0, b_0, h_0, \gamma_0, \theta, c_0, e_0) = 0.5803\theta^{0.48}; \\ g_7(c) = K(\alpha_{s0}, a_0, b_0, h_0, \gamma_0, \theta_0, c, e_0) = 0.4096 c^{0.5328}; \\ g_8(e) = K(\alpha_{s0}, a_0, b_0, h_0, \gamma_0, \theta_0, c_0, e) = 1.99 E^{0.0074}. \end{cases} \quad (15)$$

For the representative parameter of dam height h, an example of a combined graph of K_{st} approximation using one-dimensional functions $\varphi_4(x_4)$ and $g_4(x_4)$ is shown in Fig. 4.

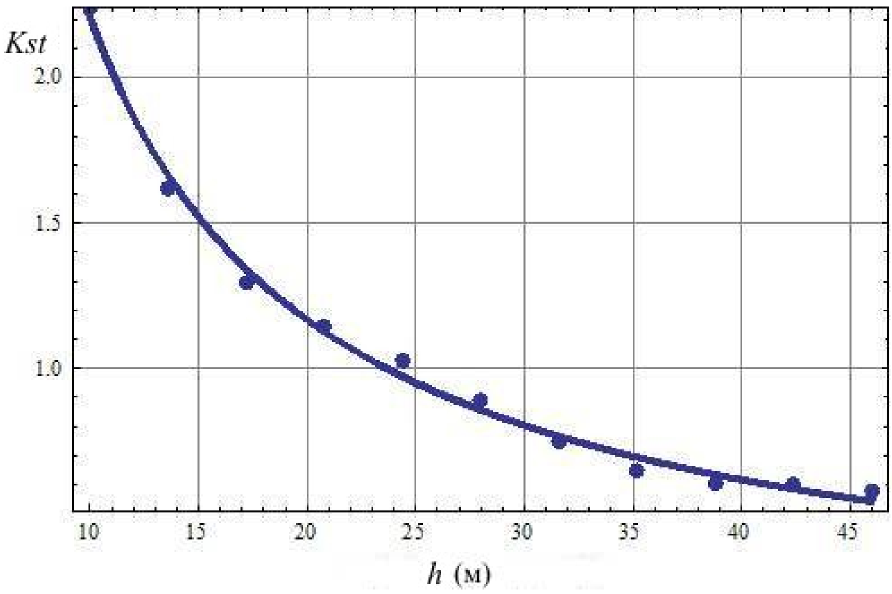


Fig.4 Combined graph of approximation K_{st} by one-dimensional functions $\varphi_4(x_4)$ and $g_4(x_4)$ for variable h

<u>Step 4</u>. Define $\phi(X)$ according to step 1:

$$\phi(x_i) = \alpha \prod_{i=1}^{n} g_i(x_i)$$

or

$$\phi(\alpha_s,a,b,h,\gamma,\theta,c,e)=\alpha g_1(\alpha_s)g_2(a)g_3(b)g_4(h)g_5(\gamma)g_6(\theta)g_7(c)g_8(E)$$

or

$$\phi(\alpha_s,a,b,h,\gamma,\theta,c,e)=\alpha\frac{b^{0.12087}\theta^{0.48}c^{0.5328}E^{0.0074}}{\alpha_s^{0.61557}a^{0.0054}h^{0.92166}\gamma^{0.6685}},\qquad(16)$$

where α is coefficient of approximation.

Step 5. Determine the function in the neighbourhood of base point M_0 from the approximate equation:

$$K_{st}(\alpha_s,a,b,h,\gamma,\theta,c,E)\approx\phi(\alpha_s,a,b,h,\gamma,\theta,c,E)=$$
$$=\alpha\frac{b^{0.12087}\theta^{0.48}c^{0.5328}E^{0.0074}}{\alpha_s^{0.61557}a^{0.0054}h^{0.92166}\gamma^{0.6685}},\qquad(17)$$

Finally, obtain the required representation of K_{st} in in the following form:

$$K_{st}=\phi(\alpha_s,a,b,h,\gamma,\theta,c,E)=\alpha\frac{b^{0.12087}\theta^{0.48}c^{0.5328}E^{0.0074}}{\alpha_s^{0.61557}a^{0.0054}h^{0.92166}\gamma^{0.6685}},\qquad(18)$$

where α is coefficient of approximation, determined according to the formula:

$$\alpha=\frac{K\left(\alpha_s^0,a^0,b^0,h^0,\gamma^0,\theta^0,c^0,E^0\right)}{g_1\left(\alpha^0\right)g_2\left(a^0\right)g_3\left(b^0\right)g_4\left(h^0\right)g_5\left(\gamma^0\right)g_6\left(\theta^0\right)g_7\left(c^0\right)g_8\left(E^0\right)}\qquad(19)$$

for the selected base point α=30.7622.

It should be noted that the complexity of the problem under consideration required a slight correction in the determination of the approximation coefficient (19). This allowed for improving the accuracy in obtaining the values of the stability coefficient of dam faces.

A check for relative errors of the obtained formulas (19) in comparison with the corresponding formulas presented in (15) showed that their magnitude does not exceed 5%.

Summary.

1. The studies conducted in determining the approximation coefficient α demonstrated that the proposed formula for its determination (4) does not always provide values that ensure minimal relative errors in the function determination.

2. Specifically difficulties in its determination arise when considering multi-parameter problems. The task of determining the safety factor of the dam is exactly this type of problem. It can be assumed that the topology of the functional surface has a folded nature, i.e., it is far from smooth. Likely, this circumstance caused the difficulties in determining the approximation coefficient value.

5. Conclusions

1. The results demonstrate the effectiveness of the proposed approach SAM to determining MM in the form of analytical formulas.

2. The obtained formula for determining the stability coefficient using power functions allows for a visual and approximate evaluation of the impact of variations in specific parameters on its value.

3. The proposed formula for the stability coefficient of the dam slopes allows for approximate assessment of the risks of the structure's instability due to certain changes in the parameter values.

4. It has been established that mathematical models can be obtained from a unified perspective through experimental research. Experimental research is understood as both physical modelling and numerical modelling.

5. Any study that can be represented as a "black box" can use the proposed hypothesis to obtain mathematical models in analytical form.

Conflict of interest

Authors state no conflict of interest.

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ПОБУДОВА МУЛЬТИПЛІКАТИВНОЇ МАТЕМАТИЧНОЇ МОДЕЛІ ГЕОМЕХАНІЧНОЇ СТІЙКОСТІ ДАМБ ХВОСТОСХОВИЩА МЕТОДОМ ПОСЛІДОВНОЇ АПРОКСИМАЦІЇ

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Анотація. Питання створення модельних інструментів моніторингу, оцінки та прогнозу безпечного природно-техногенного стану гідротехнічних споруд до яких належать укісні споруди хвостосховищ (дамби) є однією з актуальних проблем. Представлені у нормативних документах, що встановлюють вимоги до проектування, будівництва та експлуатації гідротехнічних споруд, моделі розрахунку допустимих параметрів стійкості дамб базуються насамперед на напівемпіричних моделях граничного стану. У той же час сучасний світовий досвід показує, що у практиці проектування та експлуатації гідротехнічних споруд найчастіше використовують моделі теорії твердого деформованого тіла у поєднанні з методом скінчених елементів.

Метою роботи є побудова мультиплікативної математичної моделі для визначення чутливості функції стійкості дамби від варіації її конструктивних параметрів та фізико-механічних властивостей складових з використанням методу послідовної апроксимації.

У процесі досліджень при розробці моделі апроксимація коефіцієнту стійкості здійснювалась у мультиплікативному вигляді де складові добутку є степеневі функції, кожна з яких залежить лише від одного параметру, чутливість його до варіації параметрів встановлювалась за показниками степеню функцій, коефіцієнт апроксимації або мультиплікатор виконував роль вільного параметру, що регулював адекватність параметрів точки прогнозу за виконання процедури екстраполяції. Вхідні дані для отримання моделі апроксимації коефіцієнту запасу стійкості укісних споруд створювались серією чисельних експериментів на основі геомеханічних характеристик реального об'єкту - внутрішньої дамби хвостосховища.

Науковими результатами дослідження є: побудова моделі коефіцієнту запасу стійкості на базі методу послідовної апроксимації (МПА), що дозволило не тільки отримати аналітичний вигляд критерію у околі точки, але й продовжити рішення на всю область визначення функції, з похибками, що не перевищують величин достатніх для прикладних задач геомеханіки; сформовану гіпотезу про існування представлення функцій у вигляді добутку функцій, кожна з яких залежить від одного параметра. Встановлено, що за допомогою синтезу МПА та комп'ютерних експериментальних досліджень можна отримувати сімейства детермінованих мультиплікативних математичних моделей об'єктів різних типів.

До практичних результатів дослідження належить отримана формула для визначення коефіцієнту запасу стійкості укісних систем дамб хвостосховищ, яка дозволяє наближено оцінювати ризики втрати стійкості споруди від зміни значень параметрів.

Ключові слова: чисельний експеримент, теорія чутливості, варіація параметрів, апроксимація функції, наближена оцінка, низька похибка, хвостосховище, геомеханічна стійкість дамби.